

- Lecture 11/10/21 - Section ??

- last time: Line integrals

- FTL I: If f is a function with c/s partial derivatives and C is a smooth curve parameterized by $\vec{r}(t)$ on $[a, b]$, then: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Ex: compute $\int_C \vec{v} \cdot d\vec{r}$ for $\vec{v} = \langle \sin(y), x \cos(y) + \cos(z), -y \sin(z) \rangle$
for C parameterized by $\vec{r}(t) = \langle \sin(t), t, 2t \rangle$ on $[0, \frac{\pi}{2}]$

Sol: check if FTL I holds:

① check if \vec{v} is conservative (does it satisfy Clairaut's thm?):

$$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial y}[\sin(y)] = \cos(y)$$

$$\frac{\partial}{\partial z}[v_x] = \frac{\partial}{\partial z}[\sin(y)] = 0$$

$$\frac{\partial}{\partial x}[v_y] = \frac{\partial}{\partial x}[x \cos(y) + \cos(z)] = \cos(y)$$

$$\frac{\partial}{\partial z}[v_y] = \frac{\partial}{\partial z}[x \cos(y) + \cos(z)] = -\sin(z)$$

$$\frac{\partial}{\partial x}[v_z] = \frac{\partial}{\partial x}[-y \sin(z)] = 0$$

$$\frac{\partial}{\partial y}[v_z] = \frac{\partial}{\partial y}[-y \sin(z)] = -\sin(z)$$

Partial derivatives match. So Clairaut's thm
and FTL I both hold



② compute potential function:

$$\frac{\partial f}{\partial x} = \sin(y), \quad \frac{\partial f}{\partial x} = x \cos(y) + \cos(z), \quad \frac{\partial f}{\partial z} = -y \sin(z)$$

Now $f(x, y, z) = \int \frac{\partial f}{\partial x} dx = \int \sin(y) dx = x \sin(y) + C(y, z)$

and $x \cos(y) + \cos(z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [x \sin(y) + C(y, z)] = x \cos(y) + \frac{\partial C}{\partial x}$

solve for $\frac{\partial C}{\partial y}$: $\frac{\partial C}{\partial y} = x \cos(y) + \cos(z) - x \cos(y) = \cos(z)$

Hence: $C(y, z) = \int \frac{\partial C}{\partial y} dy = \int \cos(z) dy = y \cos(z) + D(z)$

Now: $f(x, y, z) = x \sin(y) + C(y, z) = x \sin(y) + y \cos(z) + D(z)$

$$\therefore -y \sin(z) = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} [x \sin(y) + y \cos(z) + D(z)]$$

$$-y \sin(z) = -y \sin(z) + D'(z)$$

$$0 = D'(z)$$

$$\therefore D(z) = E \text{ where } E \text{ is some constant}$$

$\therefore f(x, y, z) = x \sin(y) + y \cos(z) + 0$ ↖ we chose $E=0$

③ Now Apply FTLI:

$$\int_C \vec{v} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\vec{r}(b) = \vec{r}\left(\frac{\pi}{2}\right) = \left\langle \sin\left(\frac{\pi}{2}\right), \frac{\pi}{2}, 2\left(\frac{\pi}{2}\right) \right\rangle = \left\langle 1, \frac{\pi}{2}, \pi \right\rangle$$

$$\vec{r}(a) = \vec{r}(0) = \left\langle \sin(0), 0, 2(0) \right\rangle = \left\langle 0, 0, 0 \right\rangle \quad \rightarrow$$

these came from $\vec{r}(t)$ which was given,
we just plugged in our endpoints $[0, \frac{\pi}{2}]$.

we computed this \downarrow

$$f(x, y, z) = x \sin(y) + y \cos(z)$$

- Lastly: $\int_C \vec{v} \cdot d\vec{r} = f\left(1, \frac{\pi}{2}, \pi\right) - f(0, 0, 0)$

$$= f\left(1, \frac{\pi}{2}, \pi\right) - f(0, 0, 0)$$

$$= \left(1 \left(\sin\left(\frac{\pi}{2}\right)\right) + \frac{\pi}{2} \cos(\pi)\right) - (0 \sin(0) + 0 \cos(0))$$

$$= 1 + \frac{\pi}{2}(-1) - 0 = 1 - \frac{\pi}{2}$$

$$= \boxed{1 - \frac{\pi}{2}}$$

- Recall: changing orientation of the curve (the direction we go on it) negates the curve

i.e. $\int_{-C} \vec{v} \cdot d\vec{r} = - \int_C \vec{v} \cdot d\vec{r}$

- Independence of path for line integrals of conservative vector fields.

- Prop: given a conservative v.f. \vec{v} and 2-points α, β ; we have $\int_C \vec{v} \cdot d\vec{r} = f(\beta) - f(\alpha)$ for every curve from α to β .

defined on some open region R .

- Prop: A v-field is conservative iff for all points α, β in R and all curves C, D from α to β we have $\int_C \vec{v} \cdot d\vec{r} = \int_D \vec{v} \cdot d\vec{r}$

i.e. \vec{v} is conservative precisely when it satisfies independence of paths.

- claim: If \vec{v} satisfies independence of paths, then
 define a function $f = \int_{\alpha}^{\vec{x}} \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot d\vec{r}$ (for any curve from α to \vec{x} where α is fixed in advance)

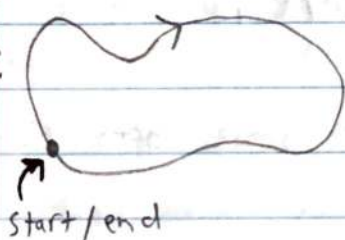
• The function f makes sense

b/c $\int_C \vec{v} \cdot d\vec{r}$ is independent of C .

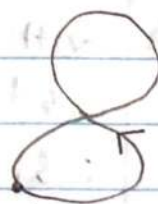
• What remains follows from the chain rule and the FTC (exercise)

Defn A simple closed curve is a curve w/o self intersection which starts and ends at the same point.

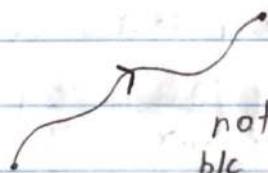
Pictures:



is an SCC



Not an SCC
b/c of self intersection



not an SCC
b/c not closed

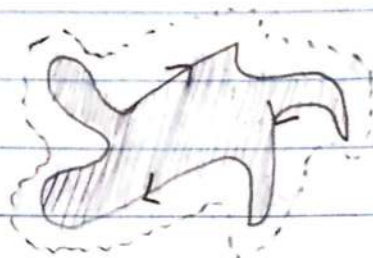
Prop: A v.f. defined in open Region R is conservative
 iff for all simple closed curves C we have
 $\int_C \vec{v} \cdot d\vec{r} = 0$



§16.4: Green's Theorem

- Idea: we want to connect some special line integrals to double integrals.

Picture:



- Idea: turn a line integral over a region cut out by an sec into a double \int .

- Suppose we have D , a closed Region in \mathbb{R}^2 with boundary of D a simple, piecewise-smooth, closed curve. If $p(x,y)$ and $Q(x,y)$ have cts. partial derivatives on some open region O containing D , then

$$\underbrace{\int_{\partial D} P dx + Q dy}_{\text{positively oriented}} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- EX] compute $\int_C x^4 dx + xy dy$ for C the positively oriented curve around the triangle w/ vertices $(0,0)$, $(1,0)$, $(0,1)$

Picture:



Parameterize the region:

$$D = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$$

$$\text{and } \partial D = C$$

\therefore by Greens thm we compute:

$$\int_C x^4 dx + xy dy = \iint_D \left(\frac{\partial}{\partial x} [xy] - \frac{\partial}{\partial y} [x^4] \right) dA$$

$$= \iint_D (y - 0) dA = \iint_D (y) dA$$

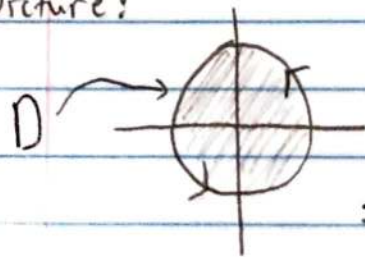
$$= \int_{x=0}^1 \int_{y=0}^{1-x} y dy dx$$

$$= \int_{x=0}^1 \left[\frac{1}{2} y^2 \right]_0^{1-x} dx = \frac{1}{2} \int_{x=0}^1 (1-x)^2 dx$$

$$\begin{aligned} \text{let } u &= 1-x \\ du &= -dx \end{aligned} \quad = \frac{1}{2} \cdot \frac{1}{3} \left[(1-x)^3 \right]_{x=0}^1 = -\frac{1}{6} ((1-1)^3 - (1-0)^3) \\ = -\frac{1}{6} (-1) = \left(\frac{1}{6} \right)$$

[Ex] compute $\int_C (3y - e^{\sin(x)}) dx + (7x + \sqrt{y^4+1}) dy$
for C the positive oriented curve around circle $x^2+y^2=16$

picture:



solution: By green's thm:

$$\begin{aligned} \int_C (3y - e^{\sin(x)}) dx + (7x + \sqrt{y^4+1}) dy \\ = \iint_D \left(\frac{\partial}{\partial x} [7x + \sqrt{y^4+1}] - \frac{\partial}{\partial y} [3y - e^{\sin(x)}] \right) dA \end{aligned}$$

$$= \iint_D [(7+0) - (3-0)] dA$$

$$\iint_D 4 dA = 4 \iint_D dA = 4 \text{ Area}(D) = 4 \pi (4)^2 = 64\pi$$